

CONTOUR INTEGRATION UNDERLIES FUNDAMENTAL BERNOULLI NUMBER RECURRENCE

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Abstract

One solution to a relatively recent American Mathematical Monthly problem [1], requesting the evaluation of a real definite integral, could be couched in terms of a contour integral which vanishes *a priori*. While the required real integral emerged on setting to zero the real part of the contour quadrature, the obligatory, simultaneous vanishing of the imaginary part alluded to still another pair of real integrals forming the first two entries in the infinite log-sine sequence, known in its entirety. It turns out that identical reasoning, utilizing the same contour but a slightly different analytic function thereon, sufficed not only to evaluate that sequence anew, on the basis of a vanishing real part, but also, in setting to zero its conjugate imaginary part, to recover the fundamental Bernoulli number recurrence. The even order Bernoulli numbers B_{2k} entering therein emerged via their celebrated connection to Riemann's zeta function $\zeta(2k)$. And, while the Bernoulli recurrence is intended to enjoy here the pride of place, this note ends on a gloss wherein all the motivating real integrals are recovered yet again, and in quite elementary terms, from the Fourier series into which the Taylor development for $\log(1 - z)$ blends when its argument z is restricted to the unit circle.

1 Introduction

An American Mathematical Monthly problem posed within relatively recent memory [1] sought the evaluation

$$\int_0^{\pi/2} \left\{ \log(2 \sin(x)) \right\}^2 dx = \frac{\pi^3}{24} . \quad (1)$$

One mode of solution depended upon integration of an analytic function around the periphery Ω of a semi-infinite vertical strip with no singularities enclosed, the quadrature having thus a null outcome¹ known in advance on the strength of Cauchy's theorem. Evaluation (1) emerged automatically by setting to zero the real part of that integral,² whereas the complementary requirement that the imaginary part likewise vanish brought into play, and successfully so, the known quadratures

$$\int_0^\pi \log(\sin(x)) dx = -\pi \log(2) \quad (2)$$

and

$$\int_0^\pi x \log(\sin(x)) dx = -\frac{\pi^2}{2} \log(2) . \quad (3)$$

With (2) and (3) in plain view, a temptation arose to provide for them, too, an *ab initio* verification, and, more even than that, to evaluate the entire hierarchy of log-sine integrals³

$$I_n = \int_0^\pi x^n \log(\sin(x)) dx \quad (4)$$

as the power of x roams over all non-negative integers $n = 0, 1, 2, 3, \dots$ ad inf. Not only was this fresh ambition, digressive and self-indulgent though it may have been, easy to satisfy via quadrature on the same contour as before, but it also exposed to view once more the fundamental Bernoulli number recurrence

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (5)$$

which is valid for $n \geq 2$ and, together with the initial condition $B_0 = 1$ and the self-consistent choice $B_1 = -1/2$, is adequate to populate the entire Bernoulli ladder, complete with null entries at all odd indices beyond $k = 1$, viz., $B_{2l+1} = 0$ whenever $l \geq 1$. Source material on the Bernoulli numbers and the related Bernoulli polynomials is ubiquitous, and can be sampled, for example, in [3-5]. References [6,7] provide a valuable overview all at once of their mathematical properties and historical genesis in com-

¹Both contour Ω , a vertical rectangle of unlimited height, and the notion of integrating an analytic function thereon so as to obtain a null result, imitate a similar ploy utilized in [2] on behalf of (2) and still further attributed there to Ernst Lindelöf.

²Two solutions for (1) were submitted by the undersigned, one involving contour integration in the manner suggested, and the other based upon a Fourier series. The Bernoulli recurrence (5) emerged as a spontaneous by-product of an ancillary, null-quadrature calculation upon that same contour Ω , initially aimed only at evaluating the log-sine integrals (4). This note embodies the content of that collateral calculation, slightly rephrased so as to highlight the newly recovered Bernoulli number sum identity.

³If that were the only goal then we should assuredly stop dead in our tracks, simply because, on the one hand, **MATHEMATICA** provides all such evaluations on demand, with great aplomb, and this even in its symbolic mode, while, on the other, a relatively painless derivation can be based upon a Fourier series, one which emerges in its turn from the power series for $\log(1-z)$ when argument z is forced to lie upon the unit circle. This Fourier series underlies in addition an essentially zinger verification of (1). All such manifold benefits of the Fourier option are sketched in an Appendix. Moreover, it goes without saying that both contour-based (14) and Fourier-based (27) evaluations of (4), even though they may be of secondary interest in the present context, do stand in complete agreement.

puting sums of finite progressions of successive integers raised to fixed positive powers. Equally valuable is online Reference [8], which cites a rich literature and covers besides a vast panorama of diverse mathematical knowledge.

Bernoulli identity (5), which is the principal object of our present concern, emerges thus by setting to zero the imaginary part of the analytic quadrature (6), below, around contour Ω , with the corresponding null value requirement on its real part providing an evaluation of the general term from sequence (4), listed in (14). No claim whatsoever is made here as to any ultimate novelty in outcome (14), which is available in symbolic form at any desired index n through routine demand from **MATHEMATICA**. Outcome (14), expressed here as a finite sum of Riemann zeta functions at odd integer arguments, continues to attract the attention of contemporary research focused upon polylogarithms [9-12]. But the formulae thus made available are subordinated in [11,12] and elsewhere to the task of evaluating a variety of dissimilar quantities, and appear to be tangled in thickets of notation. From this standpoint, formula (14) (and its identical twin (27) derived in an even more elementary fashion) may perhaps still provide the modest service of a stand-alone, encapsulated result, easily derived and easily surveyed. In particular, the canonical method of derivation evolved in [9] and repeatedly alluded to in [11,12] requires rather strenuous differentiations of Gamma function ratios, and results finally in a recurrence on the individual I_n (or else an equivalent generating function). To be sure, while the work in [9] is immensely elegant, it is at the same time immensely more intricate than either of our independent derivations culminating in (14) and (27).

On the other hand, it does appear to have escaped previous notice that the Bernoulli recurrence (5), which is ancient and foundational in its own right, should likewise reëmerge (via (16)) from the same quadrature around contour Ω when one insists that the corresponding imaginary part also vanish. And, just as is the case with (14), formula (16), too, emerges from a contact with Riemann zeta functions, but evaluated this time at even integer arguments, which latter circumstance, by virtue of the celebrated Euler connection, opens the portal to entry by the similarly indexed Bernoulli numbers. It is of course none of our purpose here to compete with, let alone to supplant in any way the standard derivations of (5). Rather, we seek merely to highlight its reëmergence in what surely must be conceded to be an unexpected setting.

We round out this note with an appendix wherein contour integration cedes place to the more elementary setting of a Fourier series on whose basis (14) is recovered yet again (as (27)) through repeated integration by parts. That same Fourier series provides moreover an exceedingly short and simple confirmation of (1), complementary to the contour integral method, an option to which allusion has already been made in Footnote 3. Of course, at this point, no further light can, nor need be shed upon (5) *per se*.

2 Null Quadratures on Contour Ω

Guided by the cited example in [2], we consider for $n \geq 0$ the sequence of numbers

$$K_n = \int_{\Omega} z^n \log(1 - e^{2iz}) dz = 0, \quad (6)$$

all of them annulled by virtue of closed contour Ω being required to lie within a domain of analyticity for $\log(1 - e^{2iz})$ in the plane of complex $z = x + iy$. Save for quarter-circle indentations of vanishing radius δ around $z = 0$ and $z = \pi$, contour Ω bounds a semi-infinite vertical strip, with a left leg having $x = 0$

fixed and descending from $y = \infty$ to $y = \delta$ (quadrature contribution L_n), and a right leg at a fixed $x = \pi$ ascending from $y = \delta$ to $y = \infty$ (quadrature contribution R_n), linked at their bottom by a horizontal segment with $y = 0$ and $\delta \leq x \leq \pi - \delta$ (quadrature contribution H_n). In what follows it will be readily apparent that the limit $\delta \downarrow 0+$ may be enforced with full impunity, a gesture whose *fait accompli* status will be taken for granted. Likewise passed over without additional comment will be the fact that no contribution is to be sought from contour completion by a retrograde horizontal segment $\pi \geq x \geq 0$ at infinite remove, $y \rightarrow \infty$.

We now find

$$L_n = -i^{n+1} \int_0^\infty y^n \log(1 - e^{-2y}) dy, \quad (7)$$

$$R_n = +i \int_0^\infty (\pi + iy)^n \log(1 - e^{-2y}) dy, \quad (8)$$

and

$$\begin{aligned} H_n &= \int_0^\pi x^n \left[\log(2) - \frac{i\pi}{2} + ix + \log(\sin(x)) \right] dx \\ &= \frac{\pi^{n+1}}{n+1} \log(2) - i \frac{\pi^{n+2}}{2(n+1)} + i \frac{\pi^{n+2}}{n+2} + \int_0^\pi x^n \log(\sin(x)) dx. \end{aligned} \quad (9)$$

Series expansion of the logarithm further gives

$$L_n = +i^{n+1} \sum_{l=1}^\infty \frac{1}{l} \int_0^\infty y^n e^{-2ly} dy = +i^{n+1} \frac{n!}{2^{n+1}} \sum_{l=1}^\infty \frac{1}{l^{n+2}}, \quad (10)$$

the interchange in summation and integration being legitimated by Beppo Levi's monotone convergence theorem, and similarly

$$R_n = -i \sum_{k=0}^n \binom{n}{k} \pi^{n-k} i^k \frac{k!}{2^{k+1}} \sum_{l=1}^\infty \frac{1}{l^{k+2}}, \quad (11)$$

in both of which there insinuates itself the Riemann zeta function

$$\zeta(s) = \sum_{l=1}^\infty \frac{1}{l^s} \quad (12)$$

at a variety of its argument values s .⁴ So armed, we proceed next to set

$$K_n = L_n + H_n + R_n = 0 \quad (13)$$

and remark that, regardless of the parity of index n , L_n *per se* is always absorbed by the contribution from the highest power y^n within the integrand for R_n . This circumstance accounts for the imminent appearance of the floor function affecting the highest value of summation index k in Eqs. (14)-(16) and (19) below.

⁴This canonical definition implies a guarantee of series convergence, assured by the requirement that $\Re s > 1$. A robust arsenal of knowledge exists for continuing $\zeta(s)$ across the entire plane of complex variable $s = \sigma + it$, with a simple pole emerging at $s = 1$.

A requirement that the real part of (13) vanish provides now the following string of valuable log-sine quadrature formulae

$$\begin{aligned} \int_0^\pi x^n \log(\sin(x)) dx &= -\frac{\pi^{n+1}}{n+1} \log(2) + \\ &+ \frac{n!}{2^{n+1}} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{(2\pi)^{n-2k+1}}{(n-2k+1)!} \zeta(2k+1), \end{aligned} \quad (14)$$

of which the first two, at $n = 0$ and $n = 1$, with the sum on the right missing, validate (2) and (3), and are in any event widely tabulated. And again, as was first stated in Footnote 3, Eq. (14) is consistently reaffirmed by **MATHEMATICA**, even when harnessed in its symbolic mode. We note in passing the self-evident fact that, unlike the corresponding prescriptions found in [9,10], formula (14) is fully explicit, needing to rely neither upon a generating function nor a recurrence, even though, naturally, such recurrence arrives at a final rendezvous with identically the same result.

A close prelude to identity (5) follows next from the coëxisting requirement that the imaginary part of (13) vanish. This requirement takes the initial form

$$\begin{aligned} -\frac{\pi^{n+2}}{2(n+1)} + \frac{\pi^{n+2}}{n+2} - \\ - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k} \pi^{n-2k} (-1)^k \frac{(2k)!}{2^{2k+1}} \zeta(2k+2) = 0 \end{aligned} \quad (15)$$

and is subsequently moulded into the shape

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k} \frac{B_{2k+2}}{(k+1)(2k+1)} = \frac{n}{(n+1)(n+2)} \quad (16)$$

on taking note of Euler's celebrated connection [3-8]

$$\zeta(2k) = (-1)^{k+1} (2\pi)^{2k} \frac{B_{2k}}{2(2k)!} \quad (k = 1, 2, 3, \dots) \quad (17)$$

allowing us to displace attention from the even-argument values of Riemann's zeta to the correspondingly indexed Bernoulli numbers B_{2k} .

3 Recurrence Reduction

Recurrence (16) is not quite yet in the desired form (5), but it is easily steered toward this goal. That process begins by noting that

$$\binom{n}{2k} \frac{1}{(k+1)(2k+1)} = \binom{n+2}{2k+2} \frac{2}{(n+1)(n+2)}, \quad (18)$$

whereupon (16) becomes

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n+2}{2k+2} B_{2k+2} = \frac{n}{2} . \quad (19)$$

Now the advance of index $2k$ in steps of two means that it reaches a maximum value $M = n - 1$ when n is odd, and one offset instead by two below n , $M = n - 2$, when n is even. At the same time the accepted null value of odd-index Bernoulli numbers starting with $B_3 = 0$ means that we are free, and self-consistently so, to intercalate all missing indices in steps of one and to entertain a common maximum $M = n - 1$, regardless of the parity of n . Altogether then, (19) reëmerges as

$$\sum_{k=2}^{n+1} \binom{n+2}{k} B_k = \frac{n}{2} , \quad (20)$$

or else

$$\sum_{k=0}^{n+1} \binom{n+2}{k} B_k = \frac{n}{2} + \left\{ \binom{n+2}{0} B_0 + \binom{n+2}{1} B_1 \right\} . \quad (21)$$

But now we find that

$$\binom{n+2}{0} B_0 + \binom{n+2}{1} B_1 = 1 - \frac{n+2}{2} = -\frac{n}{2} , \quad (22)$$

with the effect of reducing (21) to just

$$\sum_{k=0}^{n+1} \binom{n+2}{k} B_k = 0 , \quad (23)$$

which is nothing other than (5).

4 Appendix: A Fourier Series Grace Note

A somewhat more pedestrian derivation of (14) rests upon consideration of the power series

$$\log(1 - z) = - \sum_{l=1}^{\infty} \frac{z^l}{l} \quad (24)$$

along the unit circle $z = e^{i\vartheta}$. Separation into real and imaginary parts emerges as a pair of Fourier series

$$\log \left(2 \left| \sin \left\{ \frac{\vartheta}{2} \right\} \right| \right) = - \sum_{l=1}^{\infty} \frac{\cos(l\vartheta)}{l} \quad (25)$$

and

$$\left\{ \frac{\vartheta - \pi}{2}, \text{ mod } 2\pi \right\} = - \sum_{l=1}^{\infty} \frac{\sin(l\vartheta)}{l} , \quad (26)$$

of which the second is of no interest *vis-à-vis* our immediate objective. We repress all scruples henceforth as to the divergence of series (25) whenever $\vartheta = 0 \text{ mod } 2\pi$.

Repeated integration by parts *vis-à-vis* the first of these Fourier series, when multiplied by the argument power ϑ^n , advances by $\cos \rightarrow \sin \rightarrow \cos$ couplets, with end-point contributions arising only on the second beat, and the argument powers falling in steps of two.⁵ One assembles in this manner the general formula

$$\int_0^\pi \vartheta^n \log(\sin(\vartheta)) d\vartheta = -\frac{\pi^{n+1}}{n+1} \log(2) + \frac{n!}{2^{n+1}} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{(2\pi)^{n-2k+1}}{(n-2k+1)!} \zeta(2k+1) \quad (27)$$

holding good unrestrictedly for n even or odd, and agreeing in every respect with (14). The only wrinkle to notice, perhaps, is that the sequence of integrations by parts which underlies (27) terminates, at each summation index l in (25), with a term proportional to either

$$\int_0^\pi \cos(2l\vartheta) d\vartheta = 0 \quad (28)$$

in the event that n is even, or

$$\int_0^\pi \vartheta \cos(2l\vartheta) d\vartheta = 0 \quad (29)$$

otherwise. Equation (28) is of course obvious whereas (29), while equally true and welcome as such, is, at first blush, mildly surprising. All in all the derivation which underlies (14) is far smoother and less apt to inflict bookkeeping stress, even if it is (27) which seems to rest on a more elementary underpinning.

It would be truly disappointing were we not able to utilize (25) so as to give an essentially one-line, zinger-style proof of (1). This anticipation is readily met simply by squaring both sides of (25), with summation indices l and l' figuring now on its right, and noting that when, as here, both $l \geq 1$ and $l' \geq 1$,

$$\int_0^\pi \cos(2l\vartheta) \cos(2l'\vartheta) d\vartheta = \frac{\pi}{2} \delta_{l,l'} \quad (30)$$

with $\delta_{l,l'}$ being the Kronecker delta, unity when its indices match, and zero otherwise. It follows immediately that

$$I = \frac{1}{2} \int_0^\pi \left\{ \log(2 \sin(\vartheta)) \right\}^2 d\vartheta = \frac{\pi}{4} \sum_{l=1}^{\infty} \frac{1}{l^2} = \frac{\pi^3}{24} \quad (31)$$

and we are done.

5 References

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⁵In particular, this quadrature cadence provides a motivation, alternative to that previously given, as to why it is that the floor function affects the upper index cutoff $\lfloor n/2 \rfloor$ in both (14) and (27), allowing for unit growth in that cutoff only when n *per se* advances by two.

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